COUPLED OSCILLATIONS: THE REPRODUCTION AND ANALYSIS OF THE OSCILLATORY MODES OF A STRING/SPRING/MASS PENDULUM AND THE DERIVATION OF THE THEORETICAL EQUATIONS GOVERNING THE MOTION OF THE MASS

An Honors Thesis submitted by

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11 February 2009

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Introduction

For my project, I analyzed the motion of an undriven, elastic pendulum consisting of a string fixed at one end and joined to a mass via a spring. This system has two degrees of freedom: The string, spring, and mass can swing back and forth angularly, constituting the first degree of freedom; the second degree of freedom is radial and results from the elastic behavior of the spring. When these two types of motion occur simultaneously, the resulting path of the mass corresponds to one of three distinct modes of oscillation. It was the goal of this project to better understand the effects of radially- and angularly-coupled oscillations by reproducing each of these modes experimentally and determining the theoretical equations that govern the motion of the mass.

I was unable to locate any published works pertaining to this specific type of pendulum system; however, I did find work done on other compound oscillators. The most comparable example I found was the Wilberforce pendulum, which consists of a spring with a mass attached to it and also has two degrees of freedom. The mass has two vanes protruding from its sides, and as the spring is stretched and the mass is twisted, two different types of motion can be seen in the paths of the vanes. Stretching the spring away from its equilibrium position causes a bobbing motion of the mass, while twisting the mass causes an oscillation due to torsion in the spring upon release. The combined motion causes the vanes to trace out a variety of figure-eights. Figure 1.1 provides an example of a Wilberforce pendulum:

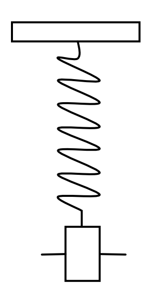
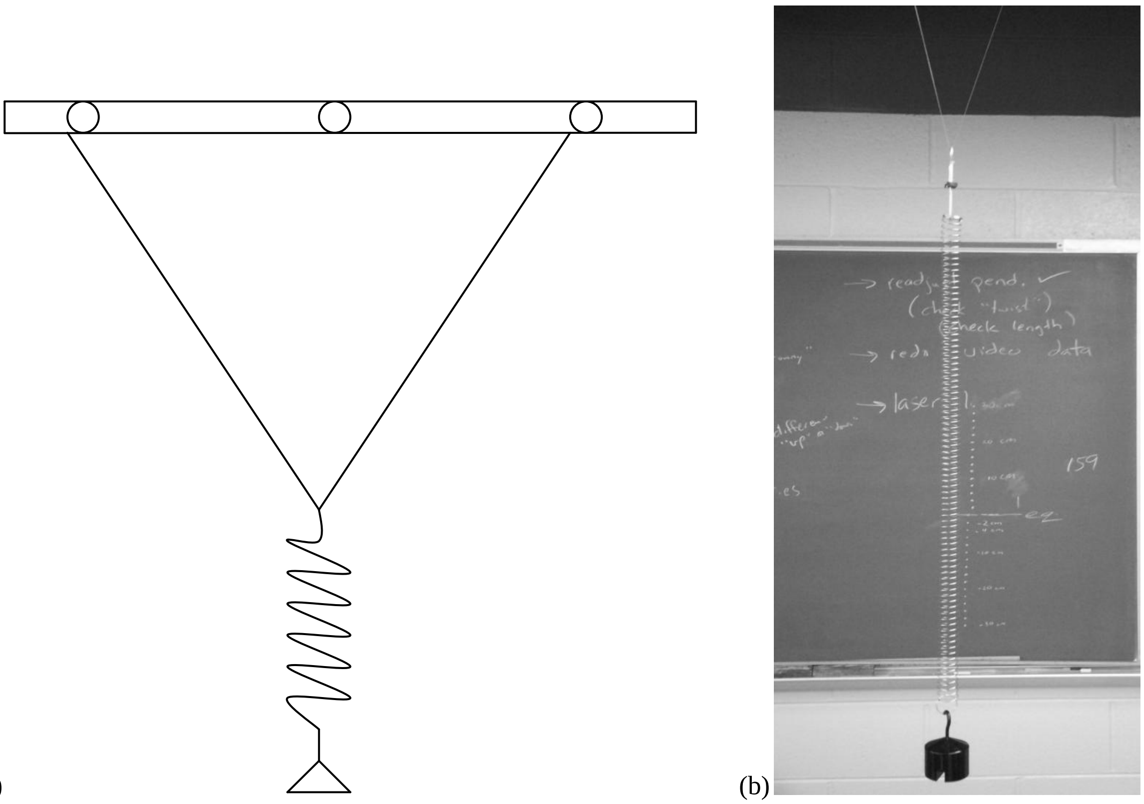


Figure 1.1: *A rough sketch of a Wilberforce pendulum. The vanes trace figure-eights as a result of the system's two degrees of freedom.*



(a) (b)

Figure 1.2: *The sketch* (a) *is not to scale. The total length of the system ended up being about 1.6 meters, which is why I could only include the lower portion in the photo* (b). *Also, these images both show the string fixed at two points rather than just one as previously indicated; this discrepancy will be explained in the next section. I used a hooked weight as the pendulum "bob."*

The experimental portion of my research occurred in one of the physics labs where I had a high ceiling and all the necessary equipment. In addition to strings, springs, and masses, I made use of a video camera, video analysis software, and data analysis software. (Detailed discussion of my experimental methods and apparatus follows this introduction.) As for the theoretical portion of my work, I took an energy conservation approach rather than a spatial coordinate approach in order to further simplify the problem of determining the equations of the mass's path. Since the total energy of the system is nearly constant, I employed Lagrange's equations of motion for conservative systems. (A detailed discussion of my theoretical work follows the section on recalibrations and corrections.)

Methods and Materials

The first step in setting up the system was to locate a suitable string, spring, and mass. In order for the system to work properly, the period of the pendulum must be twice as long as the period of the spring oscillation,[i] meaning that not just any string/spring/mass combination will suffice. The length of the string, the spring constant of the spring, and the size of the mass are all variables that affect how the system operates; so, I performed some calculations to determine exactly what combination of materials would suit my needs.

I started by using the following formula for the period of a simple pendulum:

$$T_p = 2\pi\sqrt{\frac{L}{g}},$$

where T_p is the period of the pendulum, L is the length of the string, and g is the acceleration due to gravity.[ii] For practical purposes, a pendulum with a length between 1.0 and 2.0 meters was desired. After inserting length values of 2.0 m, 1.5 m, and 1.0 m into the equation, I found that the period of a pendulum with each of these lengths would be 2.837 s, 2.457 s, and 2.006 s, respectively. (These particular string lengths served as starting points – just general estimates with no real significance other than the fact that they were reasonably long, which would serve to slow the motion of the mass, making it easier to observe and record.) Since the value of the period of the spring oscillation needs to be half that of the pendulum, the corresponding spring oscillation periods for each of the three calculated pendulum periods should have values of 1.419 s, 1.229 s, and 1.003 s, respectively. Because the value of the length of a string is easily manipulated and the physical characteristics of a spring are not, the next step was to find a spring/mass combination whose period was approximately equal to one of these three calculated spring oscillation periods. Then, the length of the string could be chosen, and its value would be within the desired range.

I obtained a set of five springs from the physics lab and determined the spring constant of each one by hanging it from a force sensor and attaching a mass to the end of it. I then pulled the spring vertically away from its equilibrium position and allowed the system to oscillate directly above a motion detector, as illustrated in Figure 1.3. I used the Logger Pro software to translate the data collected from the force sensor and the motion detector into a force vs. position graph, the slope of which gave me the spring constant of each spring. (Knowing the value of the spring constant would be important for the theoretical portion of my work. See Fig. 1.4) I knew this graph would provide me with the spring constant because Hook's Law states that the force F exerted by a spring with spring constant k that is stretched or compressed a distance of Δx away from its equilibrium position is given by the equation

$$F = -k\,\Delta x \quad \rightarrow \quad k = -\frac{F}{\Delta x}\,.$$ [iii]

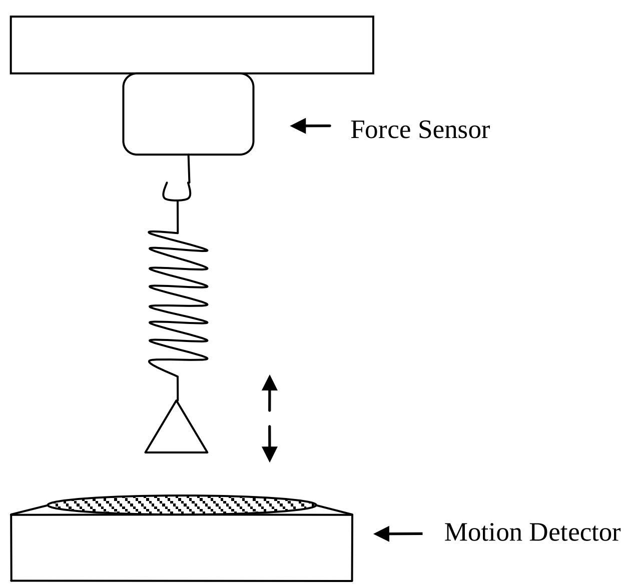


Figure 1.3: *This is a diagram of the setup for determining the spring constants of the springs.*

Next, I conducted several trials with each spring using a variety of masses to find a spring oscillation period that approximated one of those I had calculated above; I accomplished this by allowing the spring/mass systems to oscillate directly above the motion detector once again and plotting the data collected from each one in a position vs. time graph. (See Fig. 1.4)

Because a spring/mass system acts as a simple harmonic oscillator, the position vs. time graphs were all sinusoidal in nature. Upon using the Logger Pro software to fit a sine function to these graphs, I was able to determine the angular frequency, ω, for each of them. Angular frequency is related to the natural frequency by the equation

$$\omega = 2\pi f,$$

and the natural frequency is related to the period by

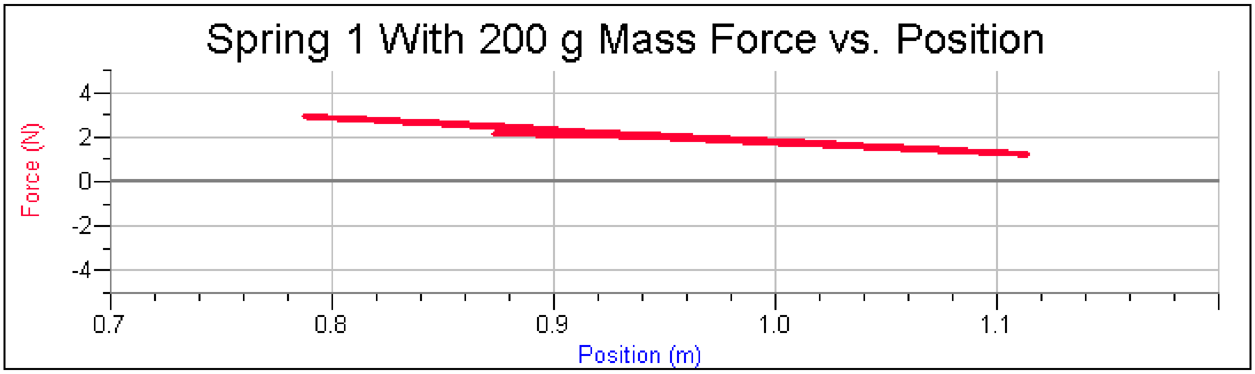
$$T = \frac{1}{f}.$$ [iv]

After some algebra and a substitution, I arrived at the equation

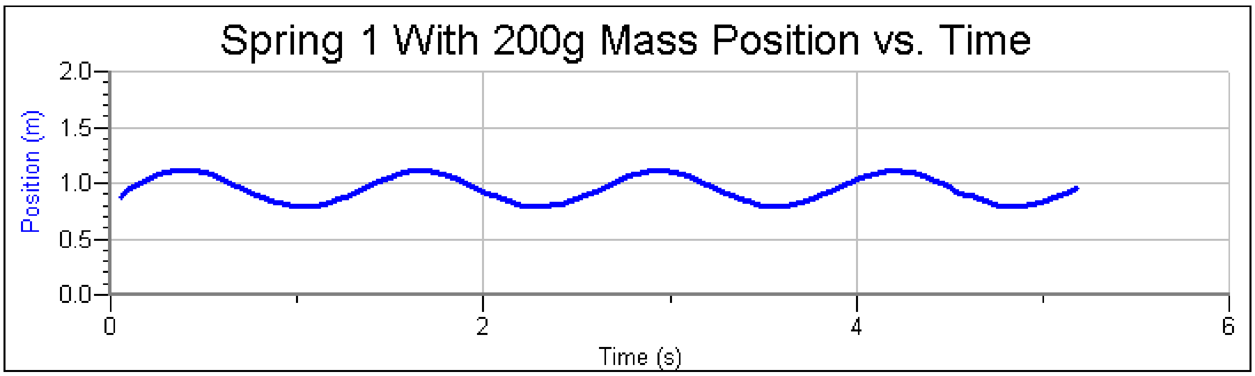
$$T_s = \frac{2\pi}{\omega}$$

for the period of a spring/mass combination. The best combination turned out to be that of Spring 1, with k = 4.816 N/m, and a mass of 200 g. I found the period of this spring/mass system to be 1.272 s, which corresponded fairly well with the spring/mass periods I had previously calculated. Using this newly found period, I then proceeded to work some of my previous calculations in reverse to determine the exact length of string I would need. Knowing that the spring oscillation period I just determined would need to be half that of the pendulum, I used the equation

$$T_s = \frac{1}{2}T_p = \pi\sqrt{\frac{L}{g}}.$$



(a)



(b)

Figure 1.4: *These graphs illustrate* (a) *force with respect to position for Spring 1 and* (b) *position with respect to time for Spring 1 with a 200g mass. The slope of the line in* (a) *is equal to the negative of the spring constant of Spring 1, while* (b) *shows that the period of this spring/mass system is just over 1 second long.*

After using appropriate values for T_s and g, I was able to determine the value of L to be 1.608 m. However, this number is not the length of the string; it is the length from the pivot point to the center of the bob. The equations I had been using thus far were for a simple pendulum with a point-mass on the end of it, but this system has a spring and a mass of non-negligible size; therefore, I had to devise a way of compensating for these discrepancies. First of all, I decided that the equilibrium length of the stretched spring needed to be included in the value for L because the spring acts as an extension of the string as far as the pendulum motion is concerned. I then concluded that accounting for the size of the mass could be easily managed by including the distance from the top of the mass to its center in the value of L, allowing me to treat the mass in my system as if it were a point located at the mass's center. Having taken these two matters into account, I determined that the length of the string should actually be L minus the sum of the equilibrium length of the stretched spring and the distance to the center of the bob. To calculate the length of the string, I simply placed the motion detector directly beneath a suspended hook and recorded the hook's position (l_1). Then, I hung the spring with the bob attached to it above the motion detector and recorded that position as well (l_2). Next, I used a ruler to measure the distance from the bottom of the bob to its center (l_3). The combined length of the stretched spring and the bob down to its center was therefore equal to

$$l_1 - l_2 - l_3.$$

The numbers I recorded from these measurements were 0.636 m for l_1, 0.126 m for l_2, and 0.016 m for l_3, which resulted in a length of 0.494 m for the combined length of the stretched spring and the bob down to its center. Subtracting this length from my total length L = 1.608 m, I found that the string itself needed to be 1.114 m long. Figure 1.5 will help the reader visualize the process of determining these values:

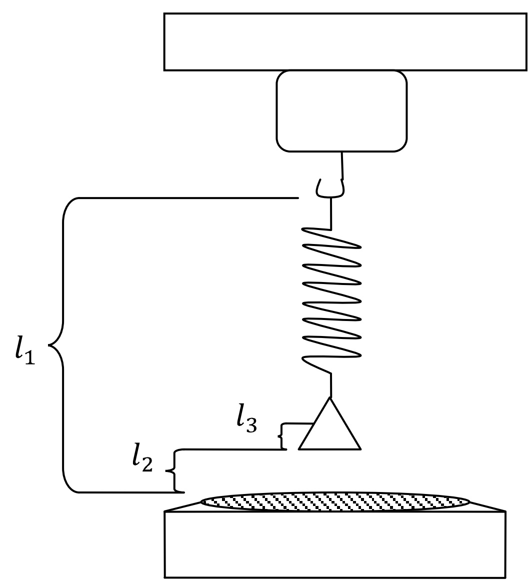


Figure 1.5: *This image illustrates the reference points for determining the values l_1, l_2, and l_3.*

After cutting the string and setting up the pendulum system, I proceeded to conduct a few trial runs. However, as soon as I got the system moving, I observed some unanticipated behavior. Rather than simply swinging back and forth as I had expected, the pendulum motion was not confined to a plane, which resulted in the bob's tracing out a star shaped path as seen from above instead of a straight line. Because the oscillatory modes I was trying to produce were only two-dimensional, I needed to come up with a way of forcing the pendulum to swing strictly back and forth in a plane. To do this, I employed a mechanism consisting of a horizontal metal bar with three evenly spaced hooks on it. I cut a 1.114 m length of string and hung it from the middle hook. Then, I tied the end of a long piece of string to one of the outer hooks and draped the rest of it over the other outer hook. Hanging a bob from the long piece of string and allowing it to pull the slack out forced the long string into a V-shape with the bob dangling from the vertex. (See Fig. 1.2) By slowly letting more slack out of the long string, I was able to lower its vertex to an equal height with the bottom end of the 1.114 m piece of string I had hung from the middle hook. At this point, I securely tied the loose end of the long piece of string to the other outer hook. This arrangement of the string proved to be much more stable; when I attached the spring/mass system to it and accelerated the mass, it forced the pendulum to move back and forth in a plane. With this problem solved, I was ready to conduct the main portion of my project's experiments.

As previously mentioned, the data collection process consisted of conducting several experimental runs and recording them with a digital video camera. I then transferred the video data to the lab computer and converted it to QuickTime format to be inserted into the Logger Pro software. Logger Pro has several useful features for dealing with video data, and I took advantage of them to locate data points and generate graphs of the bob's motion. First of all, I needed to provide the Logger Pro software with an origin and a unit length as references from which to determine the location of the bob in the video. For the origin, I selected the equilibrium position of the bob. I could not select the system's actual origin (the pivot point of the pendulum) because the system was too long to include it in the camera frame. However, this turned out to not be a problem because when I formed the equations of the bob's coordinates later, I simply subtracted the length of the system from the y coordinate. As for the reference length, I held up a meter stick in the plane of the pendulum's motion at the beginning of each recording. Then, when analyzing the video in Logger Pro, all I had to do was drag the mouse pointer from one end of the meter stick to the other and inform the software that the selected length was 1 meter. The final task was to convert the video data into a collection of coordinates that I could graph. To do this, I watched the video frame by frame via Logger Pro and located the system's center of mass with a mouse click in each frame. Logger Pro automatically interpreted the location of each mouse click as an x-y coordinate pair based on the origin and length of reference I had already given it. This frame-by-frame analysis was tedious, but "doable;" each mode analysis took 60 to 90 minutes to complete. I could then transform these coordinates into terms of r and θ to generate additional graphs. (See Fig. 2.1)

Corrections and Recalibrations

Upon returning from the summer holidays, I began to notice a problem with the pendulum. The multi-fiber string I was using to support the spring tended to get twisted at the vertex of the "V." At first I did not think this would cause a serious problem because all it effectively did was move the vertex of my string to a higher position, leaving a portion of twisted string hanging down to where the spring was connected to it. However, I noticed that the twisting and untwisting of the string introduced a third degree of motion into the system. To correct this problem, I first of all decided that using a mono-fiber for the string material would discourage twisting. I purchased some 6 lb. test fishing line and reconstructed the system, but there was still a small amount of twisting even in this line. The only other thing I could think of to change was the angle at the vertex of the string where the spring was attached, thus making the "V" wider at the top. If I increased the angle, the two halves of the string would be further apart from each other and, therefore, less likely to twist around each other. To accomplish this, I chose a new support beam for the string that would permit me to place the two ends further apart. After reconstructing the system once again using this new support beam, the twisting problem was eliminated.

Also, after switching to mono-fiber and increasing the angle at the vertex, I had to adjust the length again so that the ratio of the spring period to the pendulum period would still be 2:1. This time, rather than repeating my original method of hanging a guide line and adjusting the length of the angled line accordingly, I cut out the intermediate steps and simply let the entire system hang in equilibrium and then adjusted the length of the angled line until the vertical distance from the support beam to the center of mass of the system was 1.608 meters.

This time I also revised my method for finding the center of mass of the hanging bob to include the mass of the spring as well. Rather than simply assuming that the center of mass of the system was located at the center of mass of the bob, I instead weighed the spring and the bob and factored them both into the calculation of the system's center of mass. I ignored the mass of the fishing line in this process because it is miniscule in comparison with the spring and the bob. By treating both the spring and the bob as point masses, I was able to regard the entire mass of the system as a point mass located at the center of mass of the spring/mass system via the equation

$$m_1(x_1) + m_2(x_2) = (m_1 + m_2)(x_{cm}).$$ [v]

In this equation, m_1 and m_2 represent the masses of the spring and bob respectively. x_1 and x_2 are the locations of the center of mass of the spring (roughly the midpoint of the stretched spring) and the center of mass of the bob (a little higher than the midpoint of the bob) relative to the system's origin, and x_{cm} is the distance from the origin to the center of mass of the entire system. It turned out that the spring's mass was about 5% of the bob's mass, which made a significant difference in the location of the center of mass of the system. Thus, contrary to my initial assumptions, the new center of mass of the system was a little higher than the center of mass of the bob.

Deriving Lagrange's Equations of Motion

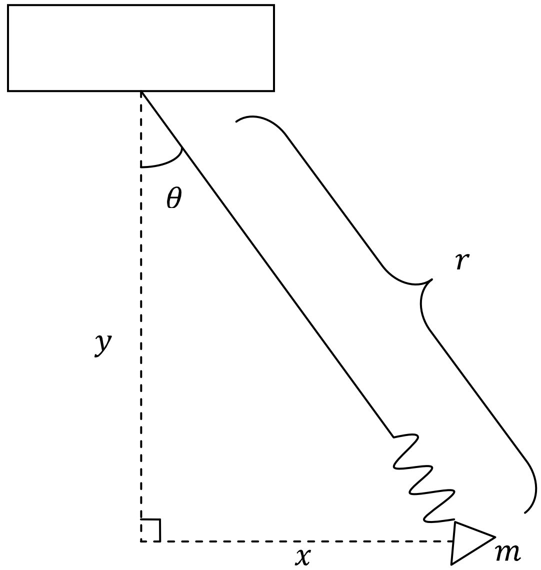


Figure 2.1: *This figure represents a still of the pendulum in motion. The variables x and y can be represented in terms of r and θ using the properties of a right triangle.*

Before discussing the different modes of oscillation, it is important to understand where the equations of motion for this system originate. The diagram in Figure 2.1 may help the reader visualize the process of determining the needed values. Because I am studying a conservative system, I chose to employ Lagrange's equations of motion, which take the energy of the system into account. The general Lagrange equation of motion for a conservative system is:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \quad \rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0,$$ [vi]

where L is the Lagrangian, which is defined as the kinetic energy of the system minus the potential energy of the system, and q_i and $\dot{q}_i$ are the position and velocity of the mass in a given direction. In the case of this system, the two degrees of freedom are the radial component and the angular component; so, polar coordinates can be used to locate the mass at any point in time. Therefore, the final form of the Lagrange equations of motion will deal with r, θ, and the radial and angular components of the mass's velocity.

From Fig. 2.1, one can see that the x-coordinate of the mass is given by $r\sin\theta$, and the y-coordinate is given by $-r\cos\theta$ using the properties of a right triangle. Taking the derivative with respect to time of each of these expressions results in the following:

$$\dot{x} = r\cos\theta\,\dot{\theta} + \dot{r}\sin\theta$$

$$\dot{y} = r\sin\theta\,\dot{\theta} - \dot{r}\cos\theta.$$

The kinetic energy is defined as

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

to take the velocity in both directions into account, and the potential energy is defined as

$$V = mgy + \frac{1}{2}k(r - r_0)^2$$

to take into account both the height of the mass and the spring potential caused by stretching the spring away from its equilibrium position (where the length of the entire system is r_0) to some new position (where the length of the entire system is r). Plugging in the values for y, $\dot{x}$, and $\dot{y}$ and then simplifying, I arrived at the following expressions for T, V, and L:

$$T = \frac{1}{2}m[r^2\dot{\theta}^2 + \dot{r}^2]$$

$$V = -mgr\cos\theta + \frac{1}{2}k(r - r_0)^2$$

$$L = T - V = \frac{1}{2}m\left[r^2\dot{\theta}^2 + \dot{r}^2\right] + \left[mgr\cos\theta - \frac{1}{2}k(r^2 - 2rr_0 + r_0^2)\right].$$

.Notice that the first term inside the brackets in the equation for T is simply the angular component of the mass's velocity squared, and the second term is the radial component of the velocity squared, which is to be expected given that the system's two degrees of freedom are in the angular and radial directions. Next, taking the partial derivative of L with respect to r, $\dot{r}$, θ, and $\dot{\theta}$ and differentiating with respect to t where appropriate, I arrived at the following:

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta - kr + kr_0$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \rightarrow \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \rightarrow \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right).$$

Finally, after plugging these expressions into the general formula, the following two equations of motion can be derived:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = mr\dot{\theta}^2 + mg\cos\theta - kr + kr_0 - m\ddot{r} = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = -mgr\sin\theta - m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right) = 0.$$

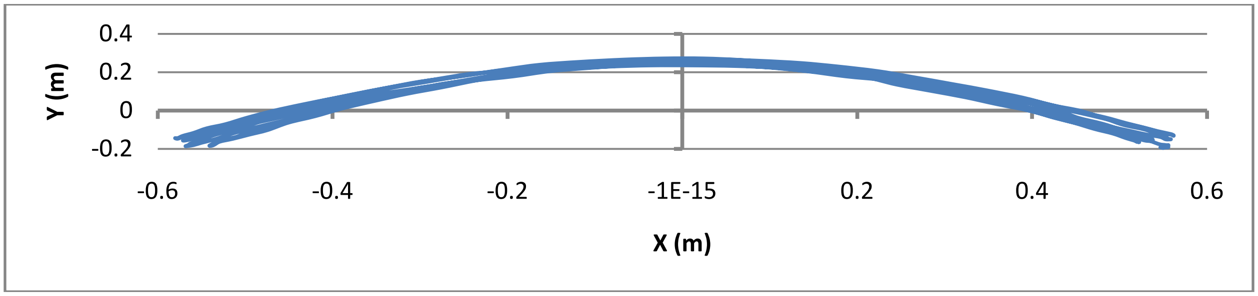
Using these equations, it is possible to describe the position and motion of the mass for any of the oscillatory modes of this system. At this point, I realized that the theoretical aspect of my project had a lot more depth to it than I had anticipated and could potentially become a new project altogether. Thus, in order to include all my experimental work as well as meet the time constraints placed upon my project, I had to abandon further theoretical exploration in favor of the empirical portion of my work. As a result, the theoretical equations I derived do not actually influence the remainder of this paper.

Oscillatory Modes

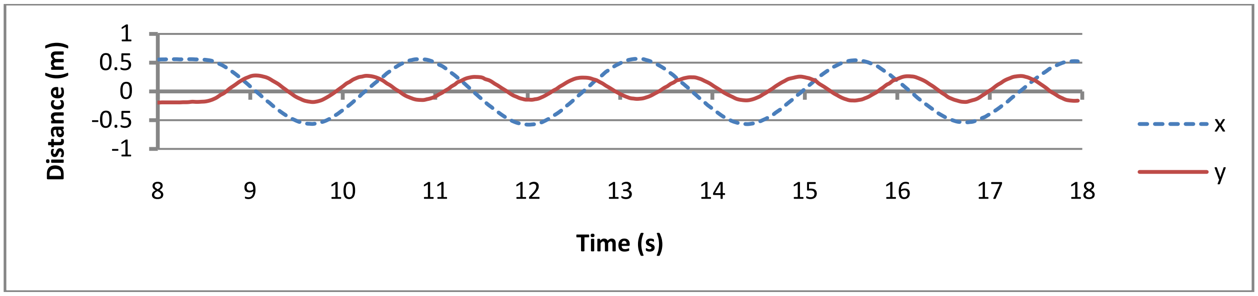
Having completed all the preliminary calculations and setup, I could now focus on reproducing and analyzing the system's modes of oscillation. The three primary modes that I identified were the Parabolic Modes, the Indeterminate Modes, and the Butterfly Mode. The Parabolic Modes, as their name implies, trace out the path of a parabola. They are produced by an initial displacement of the mass to the side and either up or down from its equilibrium position. The Indeterminate Modes are so named because they exhibit characteristics of both the Parabolic and Butterfly Modes; they shift back and forth between a series of figure-eights and a parabola over the course of their motion. To produce these modes, I had to pull the mass to the side of its equilibrium position and release it. The Butterfly Mode begins with a perfectly vertical oscillation of the mass that eventually shifts into a series of figure-eights, resulting in a path that outlines the image of a butterfly. This mode is accomplished by pulling the mass either up or down from its equilibrium position and releasing it. The end of this section consists of a brief discussion of the phase space and its contribution to the understanding of the modes.

<u>Parabolic Modes</u>

Two of the modes that the system settles into have a parabolic shape. To achieve these two modes, I simply had to pull the mass to the side and either up or down from its equilibrium position and then release it. The only difference between the two modes is that one of them traces out the path of a parabola with a positive coefficient on the x^2 term, or a concave up parabola, and the other traces out the path of a parabola with a negative coefficient on the x^2 term, or a concave down parabola. Because of the many similarities between these two modes, I will only spend time covering the latter, and the former can be assumed to behave almost identically but in the reverse direction. Figure 3.1 depicts the x-y behavior and the time evolution of the coordinates for the Concave Down Parabolic Mode as obtained from the Logger Pro software:



(a)



(b)

Figure 3.1: *These graphs indicate* (a) *the x-y behavior as well as* (b) *the time evolution of the coordinates for the Concave Down Parabolic Mode. Notice that there are multiple parabolic paths due to damping in the system. Also, we can see here that the x and y behavior are in step with each other due to the phase lock of the pendulum and spring oscillations.*

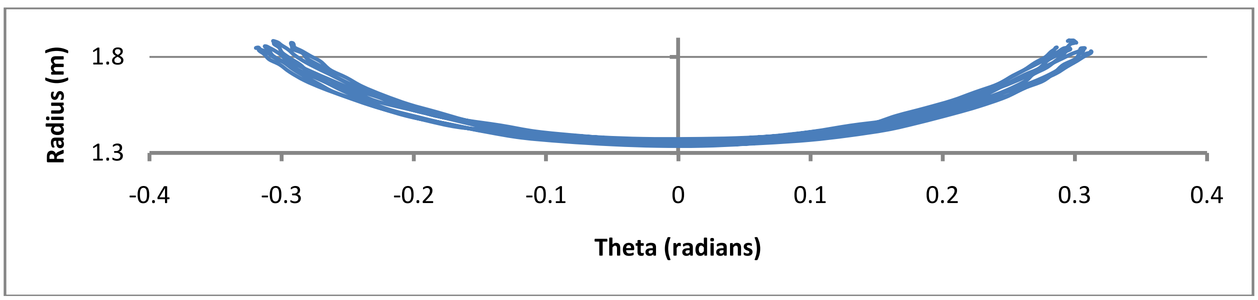
An examination of these graphs reveals that the path of the mass is an inverted parabola. This can be deduced by observing the relative positions of the local minimums and maximums for both the x and y coordinates of the mass. Whenever the x coordinate is at an extremum, the y coordinate is at a minimum, indicating that the turning points of the path are the two lowest vertical points the mass reaches. Also, whenever the y coordinate is at a maximum, the x coordinate is zero, which locates the vertex of the parabola at the highest vertical point the mass passes through.

The parabolic path of the mode is produced as a result of two factors at work in the system. First, the pendulum and the spring are phase locked with each other; that is to say that the phase relationship between their frequencies is constant. (They are locked "in step," so to speak.) Thus, the mass will trace the same path over and over, rather than alternate between different paths as is the case with the other oscillatory modes. Second, because this is a conservative system (or very nearly one at least), it obeys the law of conservation of angular momentum. As the pendulum oscillates from side to side, the spring oscillation causes the mass's distance, r, from the axis of rotation to be in a constant state of change. Because

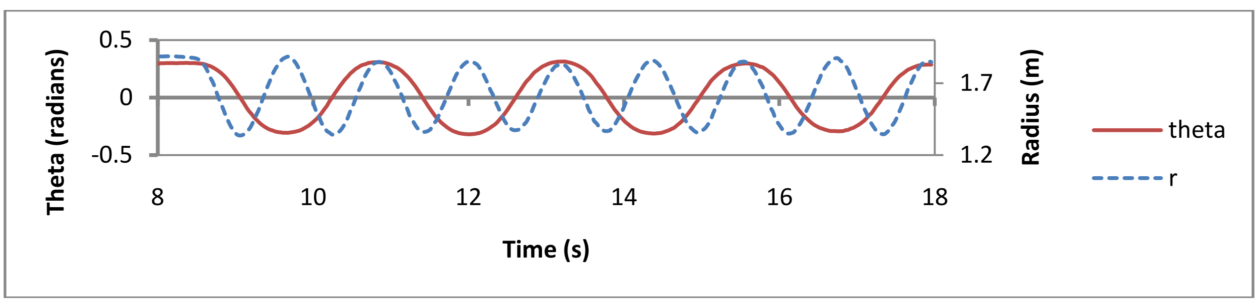
$$\vec{L} = \vec{r} \times \vec{p} \text{ and } \vec{p} = m\vec{v},$$

the only variable that can compensate for this change in r is v.[vii] Therefore, as the mass gets closer to or further away from the axis of rotation, it must speed up or slow down, respectively, to conserve angular momentum, resulting in its parabolic trajectory.

Figure 3.2 illustrates the relationship between r and θ as well as the angular and radial behavior of the mass with respect to time for the Concave Down Parabolic Mode:



(a)



(b)

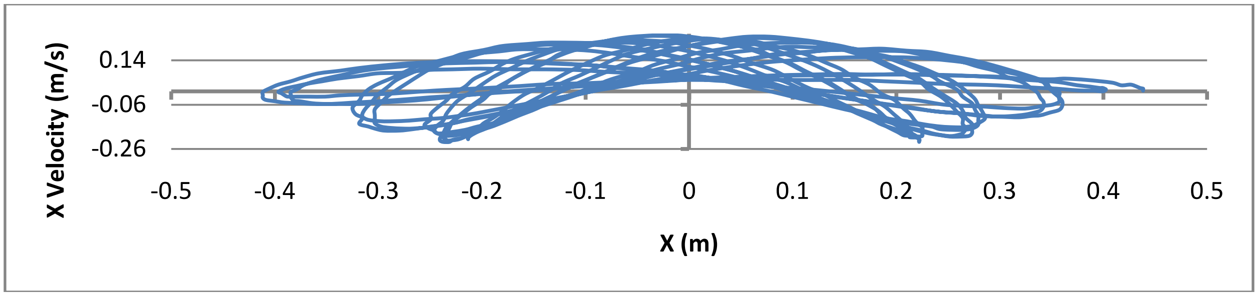
Figure 3.2: *Graph* (a) *shows the relationship between r and θ. Notice that it is concave up due to the positive relationship of the two variables. Graph* (b) *shows the time evolution of r and θ. We can still see here that the two oscillations are locked in step.*

Graph (a) in Fig. 3.2 shows how the damping in the system affects the path of the mass. Were this a perfect, frictionless system, there would only be one parabolic path visible on the graph because the total energy of the system would actually be constant during the mass's motion. However, there are multiple parabolic paths, indicating that the system's energy is changing over time. Careful study of the collected sets of data points indicates that the initial path of motion is the highest one on the graph, and each subsequent period is shifted down slightly, suggesting a decrease in energy. This energy decrease makes sense considering the presence of nonconservative forces like friction and air resistance in the system. If allowed to oscillate long enough, the system would eventually come to rest as a result of this damping, but because it has a miniscule effect over the period of interest, I was able to ignore it in my calculations.

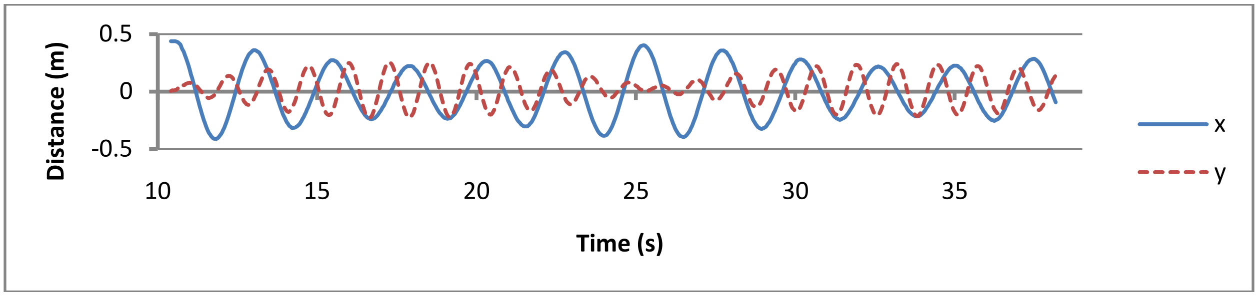
<u>Indeterminate Modes</u>

While my initial experiments led me to believe that there was only one Indeterminate Mode, I found that there were actually two. To make the system go into an Indeterminate Mode, I pulled the mass to the side of its equilibrium position and then released it. Initially, the mass began moving in an almost linear fashion. Then, it would progress to making successively taller and narrower figure-eights before tracing a narrow concave up parabola. Then, it performed all of these shapes in the reverse order to get back to its initial nearly linear motion before starting over again.

The fact that the mass traced a concave up parabola at the midpoint of the period was very interesting to me because I could think of no reason why it would be inclined to do this rather than perform a concave down parabola. Naturally, I then attempted to make it perform a concave down parabola at the midpoint of the period. In this I succeeded by pulling the mass to the side and very slightly down from its equilibrium position. This result lead me to the



(a)



(b)

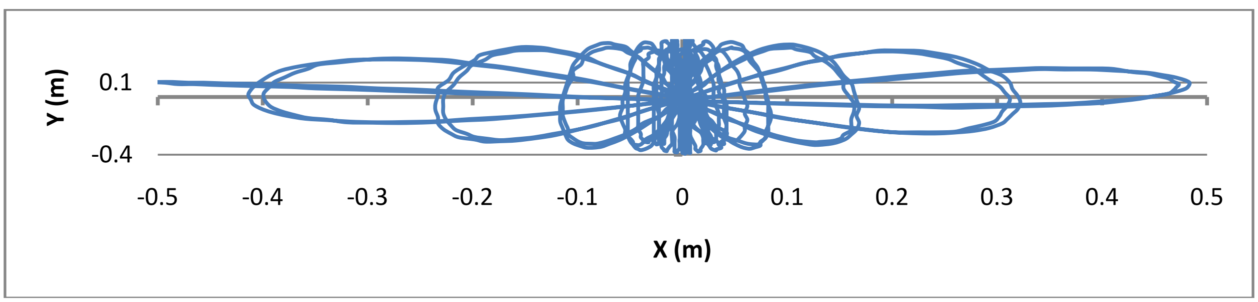
Figure 3.3: (a) *illustrates the x and y behavior of the Indeterminate Mode with a Concave Down Parabola, while* (b) *shows the time evolution of the coordinates for this mode. It is evident from* (b) *that the phase relationship between the pendulum and spring frequencies is not constant.*

conclusion that I had probably been pulling the mass to the side and slightly up in my initial experiments to produce the concave up parabola. I tried multiple times to find a point of release that would cause the parabola at the midpoint of the period to disappear, which would basically produce the Butterfly Mode shifted 90 degrees, but I could never accomplish this. Figure 3.3 above contains graphs of the x-y behavior and the time evolution of the coordinates of the Indeterminate Mode with a Concave Down Parabola at the midpoint of the period.

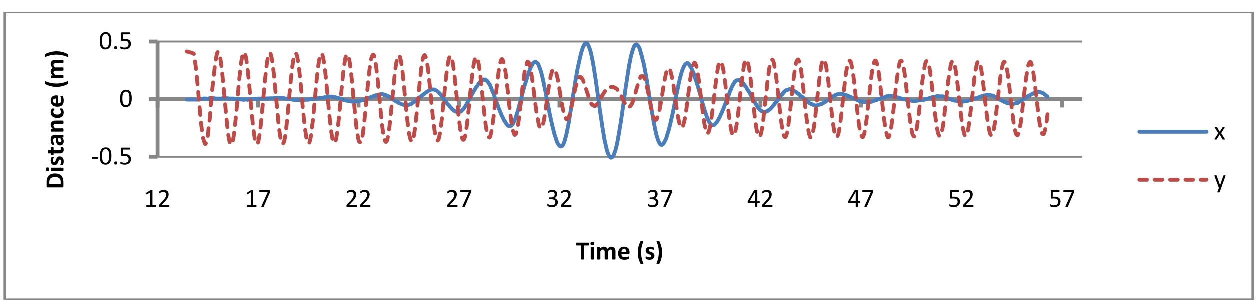
The inconsistent phase relationship of the spring and pendulum oscillations is what makes this mode possible. Initially, the two oscillations are out of phase with each other, which causes the figure-8 path of the mass. On the other hand, when the mass begins tracing out the parabola, the oscillations are in phase with each other. This pattern repeats itself over and over for the duration of the mode, resulting in the path seen in graph (a) of Fig 3.3.

Butterfly Mode

The Butterfly Mode is by far the most interesting of the modes I have studied. The behavior of this mode is due to the constant change in the phase relationship between the pendulum and the spring oscillations, which produces the following phase relationship pattern: 90 degrees out of phase, out of phase, in phase, out of phase. To initiate the Butterfly Mode, I pulled the mass either straight up or straight down from its equilibrium position and released it. (Figure 3.4 below depicts the x-y behavior and the time evolution of the coordinates for the Butterfly Mode.) The initial motion is simply a vertical oscillation of the spring/mass system behaving as if there were no string involved. Eventually, due to frictional and torsional forces at work within the system and likely a bit of human error (*i.e.* my inability to execute a perfectly vertical initial displacement), the mass began to show signs of angular motion. Over the course of the next 16 seconds, the mass traced out several figure-eights, with each successive one being



(a)



(b)

Figure 3.4: *These graphs show* (a) *the x-y behavior and* (b) *the time evolution of the coordinates for the Butterfly Mode. Notice how the near perfect symmetry of* (b) *highlights the constant change in the phase relationship between the pendulum and spring frequencies.*

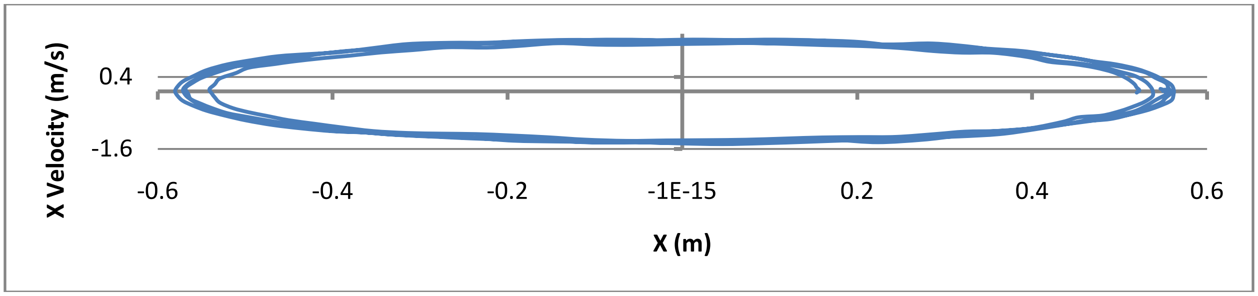
flatter and wider than the one before it. During this time, the angular motion of the mass gradually increased, and its radial motion decreased until it all but disappeared. At this point, the mass traced a very broad parabola for a few seconds before reversing the process and heading back toward a vertical oscillation.

The mass never did actually return to oscillating only vertically after the initial period of motion. Instead, as evidenced by graph (b) of Fig. 3.4, it actually traced a very narrow concave up parabola at the end of the period. Again, this could be due to an error on my part to begin with. However, it is likely the result of the dissipative forces acting within the system. Because this mode has by far the largest period of any of the three, it is much more vulnerable to damping causing major differences between successive periods. As a comparison, consider that the Parabolic Modes have a period of about 2.5 seconds, and the Indeterminate Modes have a period of about 15 seconds. The Butterfly Mode, on the other hand, has a period of 36 seconds. I experimented with this mode a lot to study the differences between its successive periods as well as to try to reduce the length of the period. I found that the more energy I put into the system (i.e. the further the initial displacement), the shorter the period of oscillation. However, due to limits in my system on the amount of energy I could safely input (I did not want to risk exceeding the elastic limit of the spring or cause the mass to jump off the spring altogether upon release), 36 seconds was the shortest period I was able to achieve.

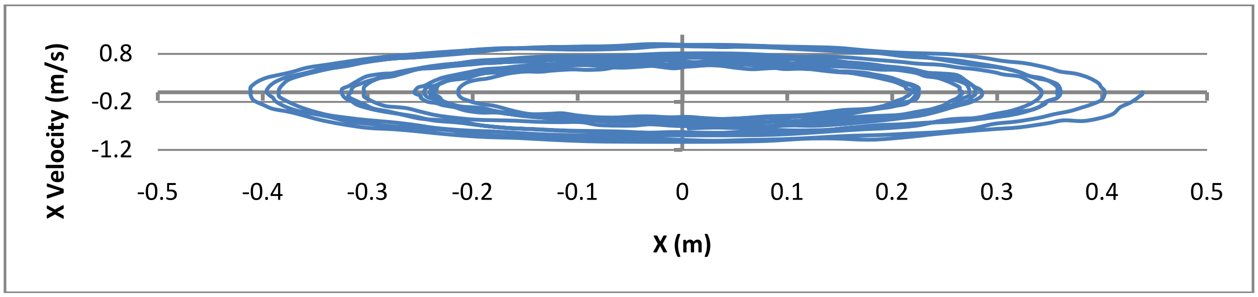
While experimenting with different initial displacements for the Butterfly Mode, I found that given a small enough displacement, the resulting path was a seemingly endless vertical oscillation of the spring/mass system. My first thought was that there may be an energy threshold that must be exceeded in order to produce the Butterfly Mode. Considering that this mode only comes about through the transfer of energy back and forth between the radial and angular oscillations, it seemed plausible to assume that an insufficient amount of energy would fail to trigger the initial transfer. Upon further investigation, I determined that the apparent anomaly was likely just the beginning of a Butterfly Mode oscillation with an extremely long period (remember that the greater the energy input, the shorter the period of oscillation, and vice versa). So, it could be that I failed to observe the pendulum for a long enough amount of time to witness the transfer of energy and the eventual production of the Butterfly Mode. Another possibility is that the initial energy input was so small that the effect of the dissipative forces was amplified enough to damp out the oscillation before it had a chance to achieve any angular motion.

Phase Spaces

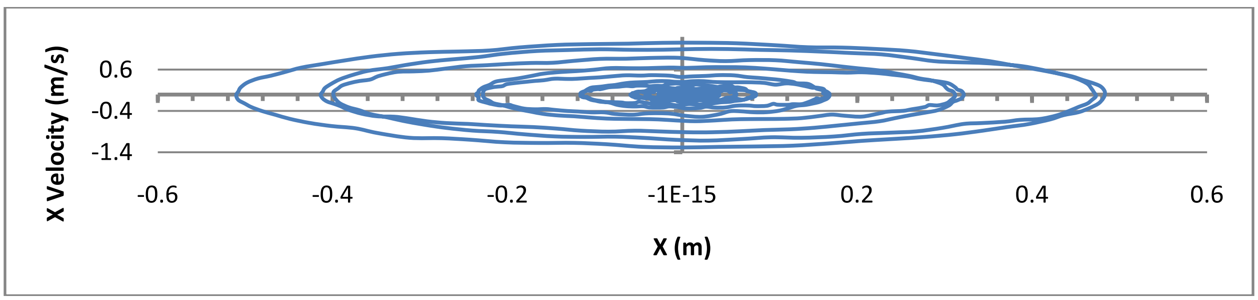
The phase space of any of the three primary oscillatory modes can be produced by graphing the velocity in the x direction, or $\dot{x}$, with respect to the x coordinate. The resulting graphs are shown below in Figure 3.5. These phase space graphs can reveal a lot about this system with regard to the kinetic and potential energy at a given point in its oscillation. For example, upon examination of the phase space for the Concave Down Parabolic Mode, one can observe that the velocity in the x direction is equal to zero whenever the x coordinate reaches an extreme value. Furthermore, because this mode is parabolic and the extreme x values are the turnaround points of the oscillation, the velocity in the y direction must also be equal to zero at these points. Therefore, we can conclude that the energy of the system is completely potential whenever the x coordinate is at an extremum. This is just one example of the many useful pieces of information one can glean from these phase spaces.



(a)



(b)



(c)

Figure 3.5: *Graphs* (a), (b), *and* (c) *show the phase spaces for the Concave Down Parabolic Mode, the Indeterminate Mode with a Concave Down Parabola, and the Butterfly Mode, respectively. These graphs can help us see the relationship between position, velocity and kinetic and potential energy for each of the modes.*

Future Directions

It is evident that the string/spring/mass system discussed herein illustrates some of the fundamental principles of physics, such as the laws of conservation of momentum and conservation of energy, and that these principles are effective tools for obtaining meaningful equations and results. However, there are some steps that could be taken to improve the experiments described in these pages in order to provide better results. This section consists of some possible future directions for this project to take. The main focus is on how to improve the results, and I also briefly consider some potential alternative setups and the impact they might have on the behavior of the system.

First of all, the issue of constructing a pendulum with the exact characteristics needed posed quite a problem, and there may be better methods of going about doing this than the ones I employed. For one thing, while the technology I had at my disposal did make the process much easier and more accurate than if I had done it alone, it still only allowed me a certain degree of precision. I could only measure confidently to three decimal places on most of my measurements. This sounds pretty precise, but there are more advanced tools that could greatly increase the precision and result in an all-around better system. Secondly, the ever-present nonconservative forces of friction and air resistance doomed the project into the realm of non-ideality from the outset. I realize that such is the case for all real systems, but there are some changes that could be made to improve in this area. For one thing, the experiment could be performed in a vacuum, which would remove the effects of air resistance from consideration. Also, the materials used to construct the pendulum could be analyzed and selected to minimize the friction at work within the system.

Another issue that probably resulted in ill effects on the modes of oscillation was the method of displacing and releasing the mass. For each experiment, I simply moved the mass with my hand to its starting point and then released it. I could not hope to move the mass to the same starting point twice with this method; nor could I ever be sure that upon release I was not imparting some initial force with my fingers in some arbitrary direction. And in the likely event that I did push the mass slightly upon release, there was no way to make sure that I was doing the same thing every time. Thus, the initial conditions for each run I performed were unique and irreproducible. One potential solution to this problem would be to use a mechanism of some sort to displace the mass and release it. This would make it possible to reproduce the initial conditions of any run, in terms of both the location and manner of release. Also, this mechanism might make it possible to determine the boundary conditions between each of the modes. My experiments only concerned reproducing each mode, but locating and possibly deriving equations to describe the boundary conditions would be a very interesting pursuit indeed.

The process of converting the video data into graphs also necessarily included some erroneous techniques. In order to produce the graphs, I first had to transfer the video I had taken to the computer in QuickTime format and then insert the video into the Logger Pro software. After doing this, I would watch each video frame by frame and locate the center of mass with a mouse click. This process was probably the source of many errors for obvious reasons. The analysis was particularly challenging when the mass was moving especially fast, resulting in a blurred image. Each mode had several hundred frames to be analyzed; so, the capacity for error was certainly non-trivial. (However, these errors did not compound on each other, and the magnitude of an individual mistake was minute.) A high speed camera would greatly simplify locating the mass in each frame by eliminating the blurring, but the number of frames to analyze would be greatly multiplied.

Another direction that might be interesting to pursue would be to construct a system similar to the one I studied, with the exception that the relationship between the spring and pendulum periods be changed. Because the modes produced by the system are a product of the ratio of the spring period to the pendulum period, altering this ratio would have a significant impact on the paths produced. In my case, this ratio was 2:1, which produced three primary modes all characterized by parabolas and/or figure-eights. On the other hand, a system with a ratio of 3:1 may produce cubic functions instead of parabolas and multiple overlapping figure-eights instead of solitary ones. A system like this may also have more than just three oscillatory modes to reproduce and examine.

While there are several matters that could be explored to improve the results of and expand the scope of this project, it was by no means a trivial undertaking. As far as I can tell, this is the first project of its kind to seek out and describe the equations of motion of this particular system for the purpose of better understanding the effects of its coupled oscillations. Having successfully reproduced each of the oscillatory modes of the system, the results of this project will lay a groundwork for future studies of this system and others similar to it.

Published works that I researched but did not specifically cite in this paper are listed along with those I did cite on the last page.[viii]

Endnotes

[i] Cordry, Dr. Sean, Professor of Physics, Carson-Newman College. Personal Interview. 7 March 2008.

[ii] Tipler, Paul A., and Gene Mosca. *Physics for Scientists and Engineers: Mechanics, Oscillations and Waves, Thermodynamics.* New York: W.H. Freeman and Company, 2004. 5th ed. p 217-247, 425-449.

[iii] See endnote ii.

[iv] See endnote ii.

[v] See endnote ii.

[vi] Fowles, Grant R., and George L. Cassiday, *Analytical Mechanics.* Belmont, CA: Brooks/Cole, 2005. 7th ed. p 417-460.

[vii] See endnote ii.

[viii] Debowska, E., et al. "Computer Visualization of the Beating of a Wilberforce Pendulum." *European Journal of Physics.* Vol. 20, No. 2, 1999. p 89-95.

Greczylo, T., and E. Debowska. "Using a Digital Video Camera to Examine Coupled Oscillations." *European Journal of Physics.* Vol. 23, No. 4, 2002. p 441-447.